代数幾何学シンポジウム記録
2008年度　pp.128-128

Abelian G-Hilbert schemes via Gröbner bases

関谷 雄飛（名古屋大学大学院多元数理科学研究科）
Yuhi SEKIYA (Graduate School of Mathematics Nagoya University)

The G-Hilbert scheme Hilb^G is defined by Ito-Nakamura. It is the irreducible component of G-fixed points of Hilbert scheme of $|G|$ points on $\mathbb{C}^n$ dominating $\mathbb{C}^n/G$ via Hilbert-Chow morphism.

A G-Hilbert scheme can be computed by using Gröbner bases.

- Fix a point p of $T = (\mathbb{C}^*)^n$.
- S : the coordinate ring of $\mathbb{C}^n$
- $I(G \cdot p) \subset S$: an ideal defining G-orbit $G \cdot p \subset \mathbb{C}^n$

Main Theorem(S—) Let G be a finite abelian subgroup of $GL(n, \mathbb{C})$. Then the following holds.

- Hilb^G is covered with affine open sets defined by reduced Gröbner bases of $I(G \cdot p)$.
- The normalization of Hilb^G is a toric variety determined by the Gröbner fan of $I(G \cdot p)$.

Notice Hilb^G is not necessary normal. Craw-Maclagan-Thomas show that Hilb^G is not normal for a subgroup $G \cong (\mathbb{Z}/5\mathbb{Z})^4$ of $GL(6, \mathbb{C})$.

For a finite small cyclic subgroup $G \subset GL(2, \mathbb{C})$, Hilb^G is the minimal resolution of $\mathbb{C}^2/G$. Hence

Corollary (Ito) For a finite small cyclic subgroup G of $GL(2, \mathbb{C})$, the toric variety determined by the Gröbner fan of an ideal $I(G \cdot p)$ is the minimal resolution of $\mathbb{C}^2/G$.

Gröbner bases can be computed by a computer, so we can examine properties of Hilb^G, for example singularity, normality and the number of torus-fixed points and so on.

Example (Hilb^G of type $\frac{1}{4}(1, 2, 3)$)

- $G = \left\langle \begin{pmatrix} \varepsilon & 0 & 0 \\ 0 & \varepsilon^2 & 0 \\ 0 & 0 & \varepsilon^3 \end{pmatrix} \middle| \varepsilon^4 = 1 \right\rangle$: a finite cyclic subgroup of $GL(3, \mathbb{C})$
- Take $p \in (\mathbb{C}^*)^3 = (1, 1, 1)$.

Then

$$I(G \cdot p) = \langle x^4 - 1, y - x^2, z - x^3 \rangle.$$

The number of all reduced Gröbner bases of $I(G \cdot p)$ is seven. So Hilb^G is covered with seven affine open sets. For example the following is the reduced Gröbner basis of $I(G \cdot p)$ with respect to weight vector $(1, 1, 1)$;

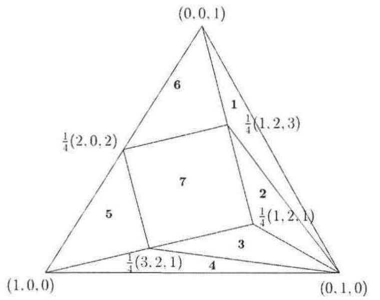
$$\mathcal{G}_7 = \{x^2 - y, y^2 - 1, z^2 - y, xy - z, yz - x, xz - 1\}.$$

The affine open set associated to $\mathcal{G}_7$ is

$$\mathrm{Spec}\,\mathbb{C}\left[\frac{x^2}{y}, y^2, \frac{z^2}{y}, \frac{xy}{z}, \frac{yz}{x}, xz\right] = \mathrm{Spec}\,\mathbb{C}\left[\frac{x^2}{y}, \frac{z^2}{y}, \frac{xy}{z}, \frac{yz}{x}\right]$$
$$\cong \mathrm{Spec}\,\mathbb{C}[X, Y, Z, W]/(XW - YZ).$$

Therefore Hilb^G is **singular**.

Moreover in this case Hilb^G is normal, so the toric variety determined by Gröbner fan of $I(G \cdot p)$ is Hilb^G. Let $N = \mathbb{Z}^3 + \mathbb{Z}\frac{1}{4}(1, 2, 3)$. We consider Gröbner fan in $N \otimes \mathbb{R}$.



This is the cross section of Gröbner fan of $I(G \cdot p)$ and corresponding toric variety is Hilb^G.

E-mail address : yuhi-sekiya@math.nagoya-u.ac.jp